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Numerical Study of the Regularized Long-Wave Equation. II: Interaction of Solitary Waves

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Numerical studies of solitary wave solutions of the regularized long-wave (RLW) equation, $u_t + u_x + uu_x - u_{xxt} = 0$, show that they exhibit true soliton behavior, being stable on collision with other solitary waves. Furthermore, arbitrary initial pulses break up into solitary waves together with an oscillating tail. Even at large amplitudes where large discrepancies might have been expected, the behavior of solutions of the RLW equation mirrors closely, both qualitatively and quantitavely, the behavior of solutions of the Korteweg–de Vries equation.

1. INTRODUCTION

This paper is the second of a two-part numerical study of the behavior of solutions of the regularized long-wave (RLW) equation [1, 8]

$$u_t + u_x + uu_x - u_{xxt} = 0. \tag{1.1}$$

In the first part of this study [3] we gave a brief discussion of the role of the RLW equation as an alternative model for nonlinear dispersive waves in many branches of applied mathematics and theoretical physics, where the classical model is the well-known Korteweg–de Vries (KDV) equation [6],

$$u_t + u_x + uu_x + u_{xxx} = 0. \tag{1.2}$$

Although the RLW equation has much nicer mathematical properties than the KDV equation, very little is known about the behavior of its solutions. This is not the case for the KDV equation, whose solutions have been studied in great detail (see [2] and the list of references therein). Bona and Smith [2] in their paper have shown that in the long-wave limit the solutions of the two models posed for the same initial data are the same.

A fundamental concept in the study of the KDV equation is the solitary wave or soliton solution, which can be written in the form

$$u(x, t) = 3c \operatorname{sech}^2 (kx - \omega t + \delta), \tag{1.3}$$

where $c = a^2$, $k = \frac{1}{2}a$, $\omega = \frac{1}{2}a(1 + a^2)$, and a and δ are arbitrary constants.

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Since the velocity of the solitary wave is $1 + c$, and hence amplitude dependent, larger solitons will overtake smaller ones and collide with them. Initial numerical studies by Zabusky and Kruskal [10] revealed the surprising result that after the collision the solitons regained their original shape, with only a change in the phase shift δ as evidence that a nonlinear collision had occurred (see Fig. 1). Similar results were found for the collison of three or more solitary waves and it was observed that, in general, any arbitrary initial pulse broke up into solitons together with a small oscillating tail. Elegant mathematical techniques to solve the KDV equation were developed by Gardner *et al.* [5] and it was found that the N-soliton solution of the KDV equation had a surprisingly simple form involving only elementary functions.

The single solitary wave solution of the RLW equation has a similar form to Eq. (1.3),

$$u = 3c \operatorname{sech}^2 (kx - \omega t + \delta), \tag{1.4}$$

where $c = a^2/(1 - a^2)$, $k = \frac{1}{2}a$, and $\omega = \frac{1}{2}a/(1 - a^2)$. It is clear that in the limit $a \to 0$ the solitary wave solutions of the RLW and KDV equations are the same. In fact, the condition $a^2 \ll 1$ corresponds to the long-wave limit of Bona and Smith [2].

We shall be studying solitons with amplitudes well outside this limit, with values of a^2 as high as $\frac{1}{2}$. (No attempt has been made to investigate the region $a \to 1$ ($c \to \infty$), since for these values of the parameters the assumptions built into both the RLW and KDV equations break down completely.) For amplitudes of this magnitude there are

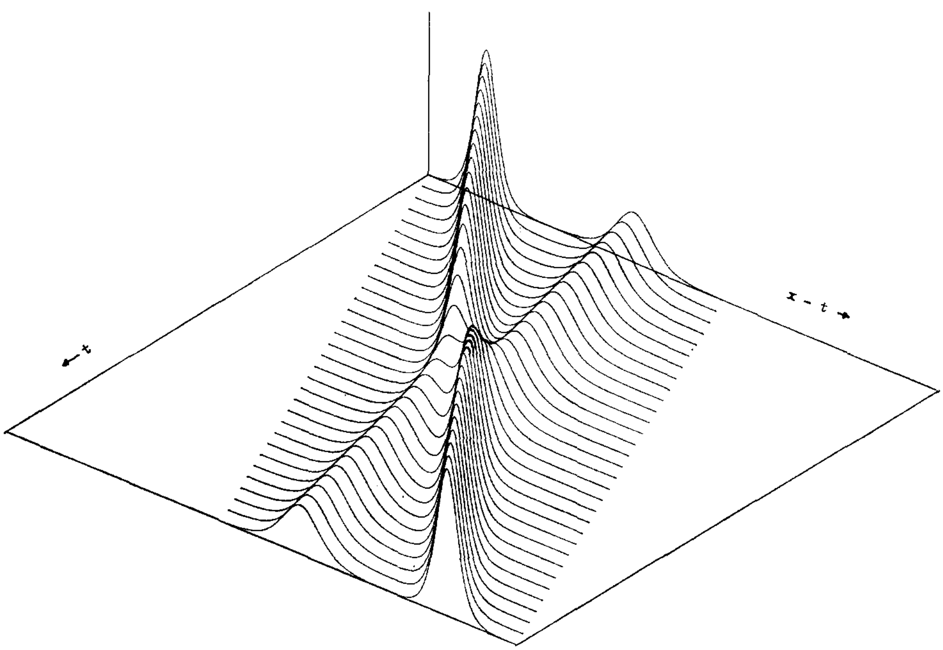

FIG. 1. Computer plot of analytic two-soliton solution of the Korteweg–de Vries equation, plotted as a function of x for different time steps. The soliton parameters are $c_1 = 0.3$, $c_2 = 0.1$.

no analytic techniques for analyzing the interaction of solitary wave solutions of the RLW equation in the same way as those for the KDV equation. The purpose of this paper is to investigate numerically whether solutions of the RLW equation, in particular, solutions with two or more solitary waves as initial values, behave in the same way as the solutions of the KDV equation. It is of interest to discover if the particle-like collisional stability of the solitary wave comes over to the RLW model and a detailed numerical investigation may help in the search for analytic techniques.

The paper is set out as follows: In Section 2 we give a brief summary of the numerical scheme developed in [3], and discuss the numerical accuracy of the results. Details of the two-soliton interaction are presented in Section 3 and the three-soliton collision and general pulse breakup are given in Sections 4 and 5. Finally, in Section 6 we summarize the results and report on the progress of investigations into the nature of the analytic form of the two-soliton solution of the regularized long-wave equation.

2. Numerical Methods

Details of various finite difference approximations for the RLW equation were discussed and analyzed in [3]. It was found that the most accurate and efficient of those schemes considered was the three level finite difference scheme

$$w_{i-1}^{m+1} - (2 + h^2)\, w_i^{m+1} + w_{i+1}^{m+1}$$
$$= w_{i-1}^{m-1} - (2 + h^2)\, w_i^{m-1} + w_{i+1}^{m-1} + \tau h(1 + w_i^{m})(w_{i+1}^{m} - w_{i-1}^{m}). \qquad (2.1)$$

Here the x and t coordinates are discretized by a grid spacing h and a time step τ. This gives the grid points $(ih, m\tau)$ with i, m integers. We use w_i^m to denote an approximation to $u(ih, m\tau)$ at the grid point (i, m). Scheme (2.1) requires a tridiagonal system of equations to be solved at each step, but since the coefficient matrix is constant it can be factorized in advance and the remaining inversion requires only two multiplications per grid point.

The truncation error of scheme (2.1) is

$$(h^2/6)\, u_{xxx}(1 + u) + (\tau^2/6)\, u_{ttt}\,.$$

Since in most practical applications $u \ll 1$ and $u_t \approx -u_x$, there is some degree of cancellation in the truncation error if $h = \tau$.

Since for a practical scheme the number of equations (2.1) has to be finite, we must introduce a right-hand boundary which was absent in the original formulation of the problem. In practice this does not cause much error if we choose a sloping boundary in the (x, t) plane and halt the program if the values of the solution close to this boundary become larger than some preset tolerance factor. In order to save computing time it is often necessary to make the left-hand boundary, originally fixed at $x = 0$, to be sloping also. Again in practice this does not cause much error as tests with a single solitary wave test solution have shown.

In making the range of x finite we are forced to chop off the tails of the hyperbolic secant (1.4). After some numerical experimentation it was found that $x = \pm 4/k$ was a reasonable compromise between accuracy and excessive calculation time. From Eq. (1.4) this corresponds to a cutoff when the function has decreased to $4e^{-8}$ of its maximum value.

Solitary wave solutions (1.4) with amplitudes in the range such that $0.05 \leqslant c \leqslant 1.0$ were considered. Since $a^2 = \frac{1}{2}$ for the largest soliton in this range, the difference between the RLW soliton (1.4) and the KDV soliton (1.3) is fairly large. Step lengths were chosen in the range $0.075 \leqslant \Delta x = \Delta t \leqslant 0.5$ with the smaller values corresponding to the larger (and hence narrower) solitons.

In testing the reemergence of the RLW solitary wave after a collision with another solitary wave we are interested in the accuracy of the numerical values of u given by the program. Tests with single solitary waves over a time corresponding to a two-soliton collision period show that the order of magnitude of the numerical errors will be at most 1 %. Since the solitary wave solutions of the RLW do, in fact, reappear after collision to an accuracy of ≈ 0.3 %, the next step is to measure the two soliton phase shift, i.e., the change in δ in Eq. (1.4) after a collision. Unfortunately, the single solitary wave test shows that the numerical errors on the δ's will be somewhat larger than the errors on the amplitude, typically in the range 3–5 %.

The relatively high accuracy of the solitary wave amplitudes in the two soliton runs is somewhat surprising. Two factors present in the two soliton collision case may be contributary effects: (i) The maximum amplitudes of the solitons during collision are smaller than the amplitude of the largest soliton before the collision and hence the truncation error in the numerical scheme is smaller; (ii) during the collision period the solitons are further from the boundary than in the single solitary wave tests and the error due to the moving boundary is less.

Typical run times for a two-soliton collision were of the order of 60 min on the Heriot–Watt Burroughs B5700 computer. An average run involved about 400–500 grid points on the (moving) x-axis and about 2000 time steps.

3. Two-Soliton Interaction

Using the three-level finite difference scheme described in Section 2, initial conditions given by the linear sum of two well-separated solitary waves of various amplitudes were inserted into the first two time levels. The initial separation was fixed according to the same prescription as in the positioning of the boundary in the single solitary wave test, i.e., two solitary waves with parameters k_1 and k_2 were started $4/k_1 + 4/k_2$ units apart. The solution was then allowed to develop in time and the amplitudes and phases of each solitary wave were calculated periodically by interpolation from the grid point values. It was found that the solitary waves passed through a strongly nonlinear interaction region and reappeared with their original amplitudes, correct to a numerical error of $\lesssim 0.3$ %. Since this error is less than the error for the single solitary waves test program this means that the RLW solitary waves behave as true solitons to within the numerical error of the calculation. The calculation was repeated with various values of the soliton amplitude such that c in (1.4) varied in the range $0.05 \leqslant c \leqslant 1.0$ and the ratio of the two soliton amplitudes, $r = c_1/c_2$ was varied in the range $1.5 \leqslant r \leqslant 10.0$. In all cases the typical collisional stability of a soliton-like interaction was observed.

A graph of RLW two-soliton collision, with $c_1 = 0.3$ and $c_2 = 0.1$, plotted for different time steps, is shown in Fig. 2. The nonlinear collision region, where the maximum amplitude is actually less than that of the largest soliton, is clearly seen, also the phase shifts of each soliton resulting from the collison. For comparision, Fig. 1 shows a

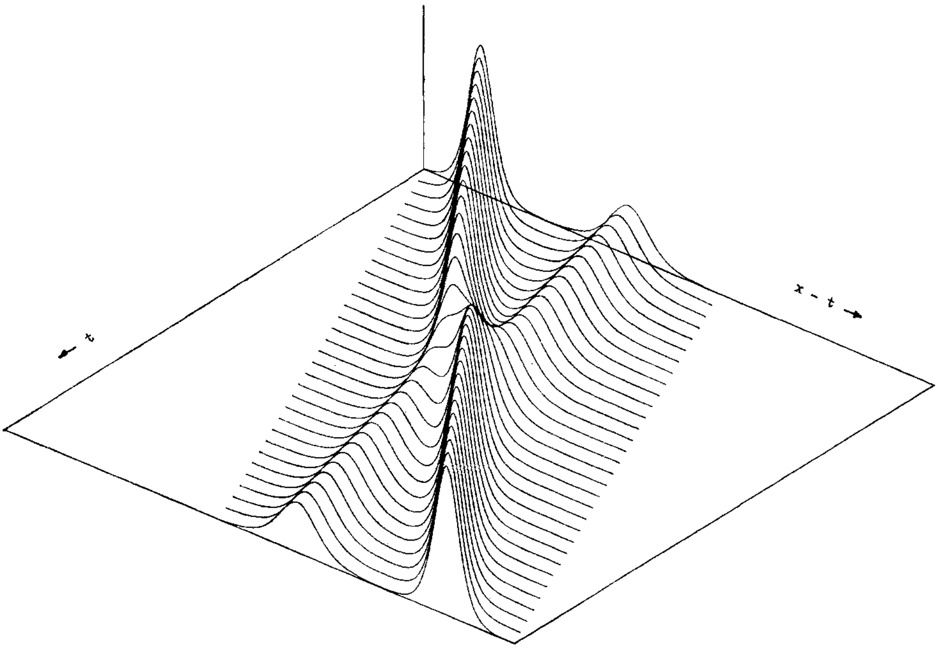

FIG. 2. Computer plot of numerical two-soliton solution of the regularized long-wave equation, with the same parameters as the KDV analytic solution in Fig. 1.

similar graph for two solitons of the same amplitudes but described by the KDV equation. Note that this two-soliton solution of the KDV equation is an analytic solution in contrast to the numerical solution of the RLW equation shown in Fig. 2. The only obvious difference between the two solutions seems to be that the KDV collision region is smaller than the RLW collision region. This is presumably due to the fact that KDV solitons are narrower than RLW solitons with the same amplitude, as can be seen from eqs. (1.3) and (1.4).

Since the RLW solitons seem to mirror the behavior of the KDV solitons, the next step is to calculate the phase shift of each soliton due to the two-soliton interaction and to compare the results with the corresponding KDV phase shifts for KDV solitons of the same *amplitude*. The KDV two-soliton phase shifts has the simple analytic form

$$\Delta\delta = \pm \ln\left(\frac{r^{1/2}+1}{r^{1/2}-1}\right), \tag{3.1}$$

where $r = c_1/c_2$ and the $-$ sign is taken for the larger soliton phase shift and the $+$ sign for the smaller soliton phase shift. Thus the phase shifts of the two solitons in the KDV case are equal and opposite and depend only on the ratio of their amplitudes.

It is interesting to note that the equal and opposite nature of the phase shifts arises from the "center of mass" conservation law for the KDV equation

$$(d/dt) \int_{-\infty}^{+\infty} xu \, dx = 0. \tag{3.2}$$

This conservation law arises from the Galilean invariance of the KDV equation [7]. No corresponding invariance property or conservation law appears to be known for the RLW equation.

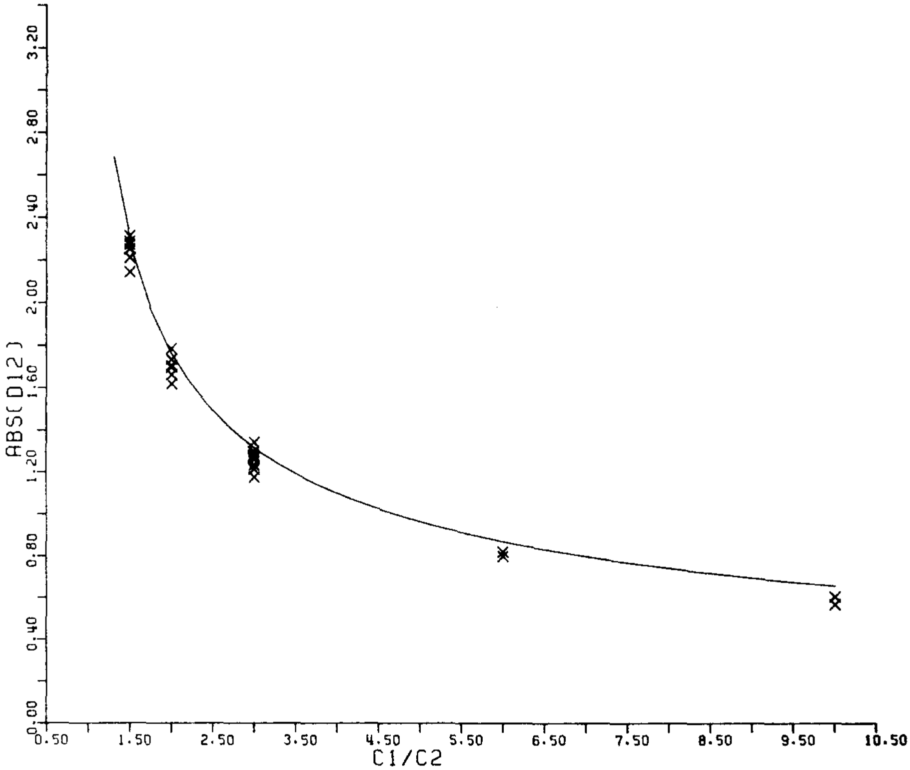

FIG. 3. Numerical values of the RLW two-soliton phase shifts, $\delta\Delta$ = D12, as a function of $r = c_1/c_2$.

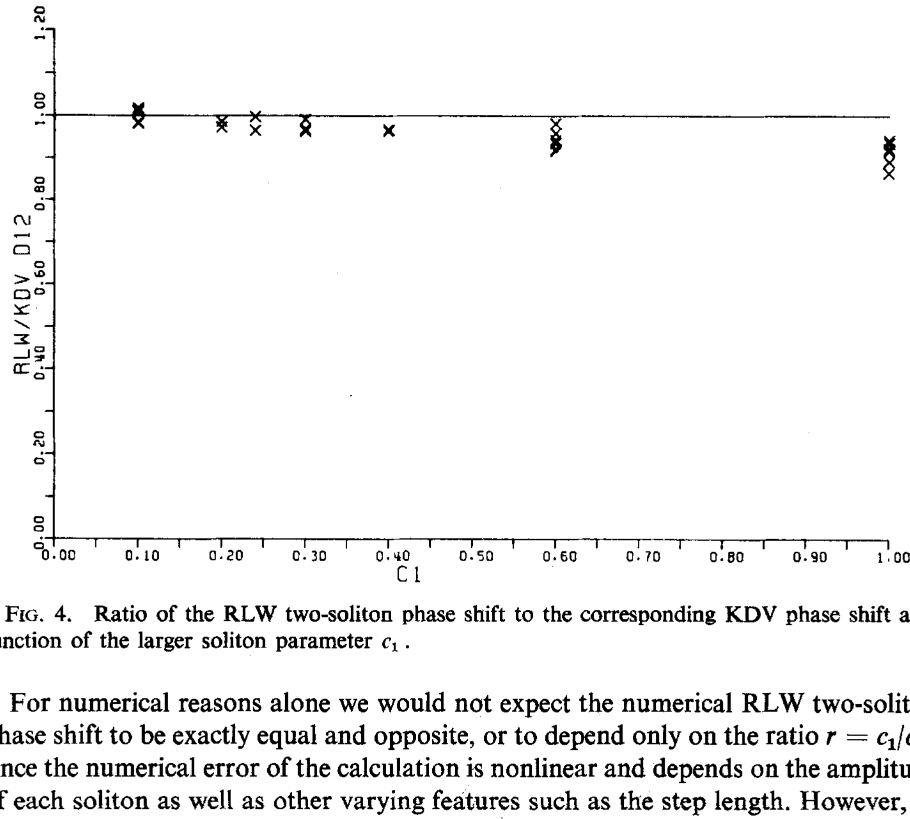

FIG. 4. Ratio of the RLW two-soliton phase shift to the corresponding KDV phase shift as a function of the larger soliton parameter c_1.

For numerical reasons alone we would not expect the numerical RLW two-soliton phase shift to be exactly equal and opposite, or to depend only on the ratio $r = c_1/c_2$, since the numerical error of the calculation is nonlinear and depends on the amplitude of each soliton as well as other varying features such as the step length. However, to within these rather large phase shift errors (3–5 %), the RLW numerical two-soliton phase shifts were equal and opposite, depended only on r, and had the same numerical value as the KDV phase shift (3.1) for both different ratios $r = c_1/c_2$ and varying values of c_1 for fixed r considered. Figure 3 shows a plot of the RLW phase shifts as a function of r compared to the KDV phase shifts. In Fig. 4 the ratio of the RLW phase shift to the corresponding KDV phase shift is plotted as a function of the largest soliton amplitude (c_1) in the collision. Note that the comparison is with KDV solitons with the same *amplitudes* as the RLW solitons; had we compared the solitons with the same values of the *wavenumbers* ($\frac{1}{2}a$ in Eqs. (1.3) and (1.4), the agreement would have been much worse. Despite the statistical fluctuation of the errors, there seems to be a small diverging trend between the KDV and RLW results for large values of c_1. This is hardly surprising as these values are a long way from the long-wave limit for which the two equations are known to agree.

4. THREE-SOLITON INTERACTION

Since analytic solutions of the KDV equation exist which describe the collision of an arbitrary number N of solitons, it is of interest to see if our numerical results for the RLW equation with $N = 2$ extend to higher values of N. We consider only the case $N = 3$.

In analogy to the two soliton interaction, initial conditions for the numerical scheme are chosen to give three solitons on a collision course. Care must be taken to chose the initial positions of the solitons such that a simultaneous collision occurs, otherwise we would be merely creating a series of two-solition collisions. However, the theory in the KDV case reveals rather surprisingly that the KDV soliton phase shift in an N-soliton collision is merely a linear sum of the appropriate two-soliton phase shifts even though the solitons collide simultaneously.

The numerical results of the three-soliton collision in the RLW case shows that the "collisional stability" observed in the two-soliton collision applies equally well in the three-soliton case. A measurement of the soliton phase shifts revealed agreement with the KDV analytic phase shift formula to within 5 %, the same order as the numerical error. A graph of the three-soliton collision is shown in Fig. 5.

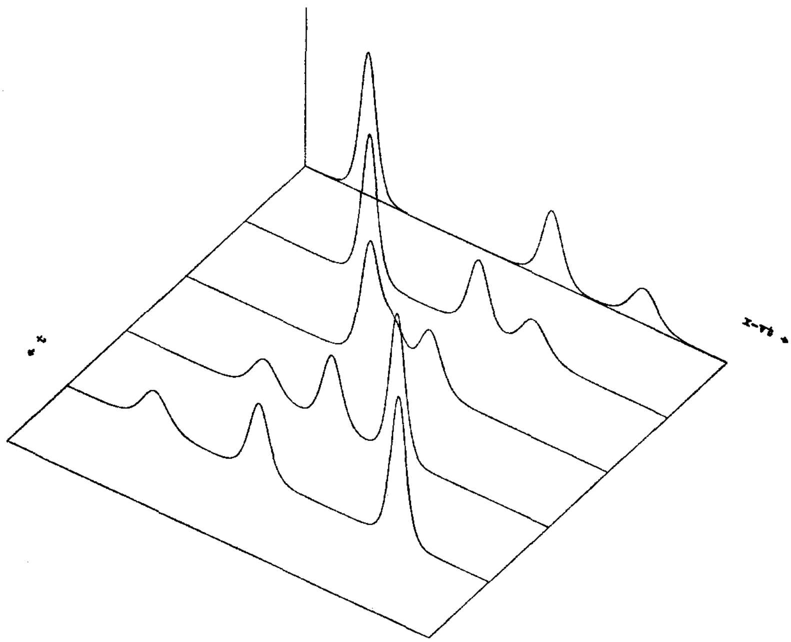

FIG. 5. Computer plot of the three-soliton numerical solution of the RLW equation, seen in a center of mass frame,where v is the velocity of the center of mass of the three solitons. Soliton parameters are $c_1 = 0.6$, $c_2 = 0.3$, $c_3 = 0.15$.

In both the two- and three-soliton collisions there is no evidence for the formation of any oscillating tail. This is hardly surprising since the solitons are stable on collision and hence they cannot lose any energy during the collision since the total energy of the system is conserved, to the order of accuracy of the numerical scheme.

5. General Initial Pulses

The inverse scattering method applied to the general initial value problem for the KDV equation [5] shows that an arbitrary initial pulse will split up into a number of solitons together with an oscillating tail. The inverse scattering problem involves solving a Schrödinger equation with the initial pulse corresponding to the "potential well;" the discrete spectra correspond to solitons and the continuous spectra correspond to the oscillating tail.

Numerical runs with arbitrary initial pulses for the RLW equation show the same qualitative behavior. In order to test the quantitative theory (and the numerical scheme) to their limit, some runs were done with a square wave pulse as initial data. This represents a complete breakdown of the "long-wave limit" assumption made in deriving the RLW equation and also a breakdown in the smoothness assumed in setting up the finite difference scheme.

The theory for the KDV equation [9] shows that for a square wave of amplitude A and width l, the number of solitons produced is the largest integer $N \leqslant S/\pi + 1$ where

$$S = (A/6)^{1/2} l. \tag{5.1}$$

The eigenvalues are solutions of the equations

$$\sin \lambda = \pm(2\lambda/s), \tan \lambda < 0 \tag{5.2}$$

or

$$\cos \lambda = \pm(2\lambda/s), \tan \lambda > 0. \tag{5.3}$$

For each of the N solutions of these equations λ_i we have a soliton with the constant c_i given by

$$c_i = 4A(1 - (2\lambda_i/s)^2). \tag{5.4}$$

For a typical RLW run we chose $A = 0.9$ and $l = (3\pi/2)(6/A)^{1/2} = 12.1673$ to give $s = 3\pi/2$ which would give $N = 2$ *if* the KDV theory also applied to the RLW equation.

The resulting numerical solution at $t = 37.5$ is shown in Fig. 6. The leading solitary wave is clearly distinguishable, with an amplitude corresponding to $c_1 = 0.4311$, and a second solitary wave with an amplitude corresponding to $c_2 = 0.1206$ appears to be separating out from the oscillating tail. The KDV theory predicts $c_1, c_2 = 0.4716$, 0.1372, so the agreement is quite good considering the difficulties introduced by the initial square wave profile. No great accuracy can be claimed for the size and structure of the rapidly oscillating tail since the wavelengths of the oscillations are comparable with the grid spacing h. Smoother initial conditions also give rise to solitary waves and an oscillating tail but the oscillations are not as violent as in the square wave case. The response of the numerical scheme to high frequency oscillations is clearly important and deserves further study.

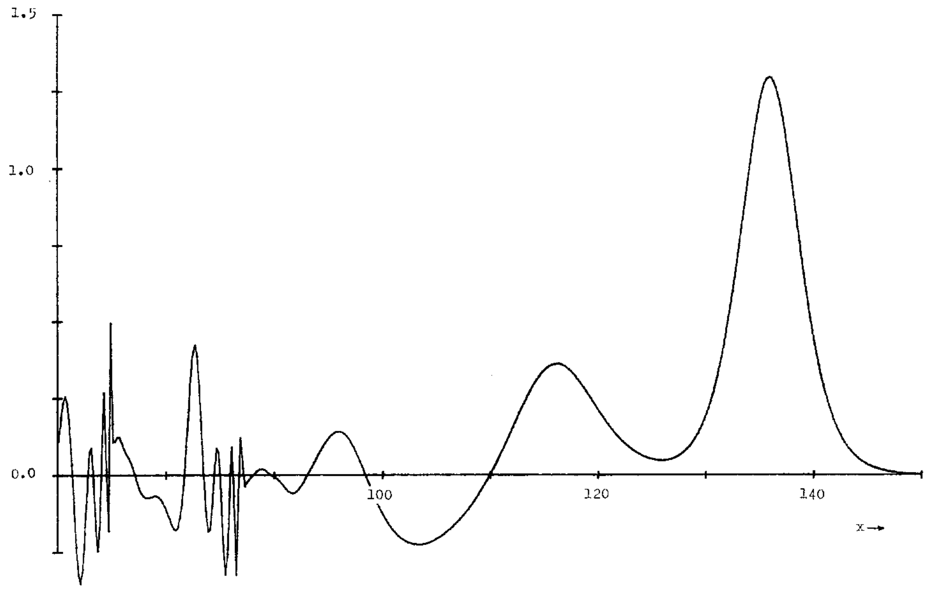

FIG. 6. Plot of the numerical solution of the RLW equation at $t = 37.50$ for a general initial pulse: a square wave $u(x, 0) = 0.9$ for $75 < x < 87.17$.

6. SUMMARY

The numerical results described in Sections 3–5 show clearly, within the limits of numerical error, that the qualitative behavior of solutions of the RLW equation is the same as the behavior of the KDV equations for initial data, which are zero outside some finite range of the x-axis.

This agreement holds even for large amplitudes for which the long-wave limit does not hold. Furthermore, the agreement is also quantitatively close, with only some evidence for a slight divergence between the two sets of two-soliton phase shifts at the higher soliton amplitudes. Further work is necessary to improve the numerical methods to study more closely the difference between the two solutions.

In view of the numerical evidence for multisoliton solutions of the RLW equation, it is natural to investigate where the RLW equation has an analytic N-soliton solution expressable in closed form as a function of elementary functions, as in the case of the KDV equation. J. D. Gibbon and the present authors [4] have studied this problem, but without success. The one and two-soliton solutions of the KDV equation can be written as

$$u = 2(\partial^2/\partial x^2) \ln f, \tag{6.1}$$

where

$$f = 1 + \exp(\theta_1) \tag{6.2}$$

for the one-soliton solution, and

$$f = 1 + \exp(\theta_2) + \exp(\theta_2) + \left(\frac{a_1 - a_2}{a_1 + a_2}\right)^2 \exp(\theta_1 + \theta_2) \tag{6.3}$$

for the two-soliton solution, with

$$\theta_i = a_i x - a_i(1 + a_i^2)t + \delta_i . \tag{6.4}$$

Since the one-soliton solution of the RLW equation (1.4) can be written as

$$u = 2(\partial^2/\partial x \partial t) \ln f, \tag{6.5}$$

where

$$f = 1 + \exp(\theta_1) \tag{6.6}$$

and

$$\theta_1 = a_1 x - a_1(1 - a_1^2)^{-1}t + \delta_1 , \tag{6.7}$$

it was hoped that the corresponding version of Eq. (6.3), possibly with higher terms added, would turn out to be the required two-soliton solution. However, no such solution of this form has been found and it seems unlikely that the RLW two-soliton solution, if it exists, has the same form as the KDV two-soliton solution.

A computer-produced 16-mm cine film of numerical solutions of the RLW equation has been produced at the SRC Atlas Computing Laboratory by one of the authors (JCE). The film shows clearly the two- and three-soliton collisions and the square wave pulse breakup described in previous sections.

Notes added in proof. Since this paper was written, a letter describing briefly some numerical experiments on the RLW and other equations has been published (Kh. O. Abdulloev *et al.*, *Phys. Lett.* **56** A (1976), 427–428). These authors claim a very small ($\sim 10^{-3}$) oscillating tail is produced when two very large (~ 10) solitary waves collide. The significance of this effect is under investigation and will be reported on in a later paper.

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